NONSTEADY TEMPERATURE FIELDS IN CHANNEL FLOW

R. A. Ivanov

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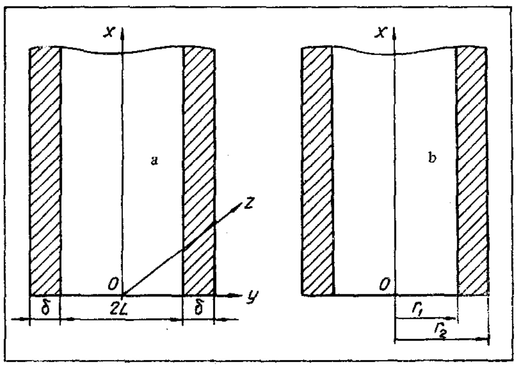
A solution is presented for the problem of the nonsteady temperature field in a system consisting of a metal and a moving heat carrier. Flow in symmetrical plane (and cylindrical) channels is considered. The problem is solved by an operational method using a Laplace-Carson functional transform.

In solving various thermophysical problems, it is often necessary to investigate the nonsteady temperature fields in a system comprising a metal and a moving heat-transfer agent for given conditions at the inside and outside channel walls. This necessity arises, for example, in investigating the dynamic processes in heat-exchange equipment and connecting channels.

Consider a plane symmetrical channel (figure, a) through which flows a heat-transfer agent whose initial temperature state is characterized by the zero temperature distribution, while the outer surface is ideally insulated. The geometric characteristics of the channel are as follows: $0 \le x < +\infty$; $-\infty < z < +\infty$; $-L - \delta \le y \le L + \delta$; the thickness of the metal wall is equal to δ.

If at a given instant there is a sudden change, for example, an increase, in the temperature of the heat-transfer agent at the channel inlet, it is seen that, as the heat-transfer agent moves along the channel, it heats the metal of the walls.

We assume that: the conductive heat transfer along the channel is negligibly small; the specific weight and mass specific heat of the heat-transfer agent, the thermal conductivity of the metal, the coefficient of heat transfer from the heat-transfer agent to the metal, and the rate of flow of the heat-transfer agent in the channel are constant throughout the transient process; the amount of heat transferred from the heat-transfer agent to the metal is proportional to the coefficient of heat transfer from the heat-transfer agent and the inner surface of the channel wall; the temperature of the heat-transfer agent is averaged over the channel cross section (this assumption is valid for highly turbulent flows); the temperature fields of the heat transfer agent and the metal in any section z = const coincide.



Plane and cylindrical symmetrical channels: a) plane; b) cylindrical.

If now, in view of the symmetry of the problem, we consider half the channel, locating the coordinate origin at the point x = 0, y = L, z = const and specify at the inlet a constant perturbation, equal to unity, with respect to the temperature of the heat-transfer agent, then the nonsteady temperature fields in the heat-transfer agent and the metal are mathematically described by a first-order linear equation—the equation of heat propagation in the heat-transfer agent [1]:

$$w\frac{\partial\theta}{\partial x}+\frac{\partial\theta}{\partial t}=-\frac{\alpha}{\gamma LC}\left(\theta-T|_{y=0}\right) \tag{1}$$

with the following boundary conditions:

$$\theta|_{x=0}=1,\ \theta|_{t=0}=0 \tag{2}$$

and by the linear equation of heat conduction with boundary conditions of the third kind [2]:

$$\frac{\partial T}{\partial t}=a\frac{\partial^2 T}{\partial y^2}; \tag{3}$$

$$\left.\begin{aligned}&-\lambda\left.\frac{\partial T}{\partial y}\right|_{y=0}=\alpha\left(\theta-T|_{y=0}\right),\\&\left.\frac{\partial T}{\partial y}\right|_{y=\delta}=0,\ T|_{t=0}=0.\end{aligned}\right\}. \tag{4}$$

Given this description of the heat-transfer process in the channel, it is possible to distinguish a series of characteristic times: a) the time of propagation of the input signal along the channel, τ_W (determined by the flow velocity of the heat-transfer agent); b) the time of heat transfer from the fluid to the metal, τ_α (determined by the coefficient of heat transfer from the heat-transfer agent to the metal); and c) the time of heat propagation in the metal, τ_λ (determined by the thermal conductivity of the metal).

In a number of cases, depending on the relationship between these times, the heat-conduction equation is eliminated from the mathematical model formulated above.

These cases are:

1) τ_α, $\tau_\lambda \to 0$; $\tau_W \le M$, where M is a certain constant. In this case, the metal of the channel is combined with the heat-transfer agent.

2) $\tau_W \to 0$; $\tau_\alpha \leq M$; $\tau_\lambda \leq N$, where N is a certain constant. If the entire dynamic process in question is of only slight duration, the heat conduction equation can be eliminated from the mathematical model.

If the characteristic times are equal, or τ_W is sufficiently large (as compared with τ_α and τ_λ), the heat-conduction equation cannot be neglected. In this case, an approximate solution is obtained, for example, by dividing the wall metal into a number of regions, with given temperature distribution, i.e., the heat-conduction equation is replaced by a system of ordinary differential equations.

In all the cases considered above, determining the corresponding error is quite a complicated matter; therefore, there is some interest in finding an exact analytic solution for subsequent comparison with the simplified models.

There follows a solution in quadratures giving the nonsteady temperature fields for a heat-transfer agent flowing in a plane or cylindrical symmetrical channel.

Plane symmetrical channel. The nonsteady temperature fields of the heat-transfer agent and the metal for a plane symmetrical channel are described, with the given assumptions, by Eqs. (1)–(4).

The solution of system (1)–(4) is found by means of a Laplace-Carson functional transform. In the transform region, it has the solution

$$\bar{\theta}(x, p) = \exp\left\{-\frac{px}{w}\right\} \times$$

$$\times \prod_{n=1}^{\infty} \exp\left\{-A_n \frac{kx}{w}\right\} \prod_{n=1}^{\infty} \exp\left\{\frac{1}{a_n(p+B_n)}\right\}, \tag{5}$$

$$\bar{T}(x, y, p) = \bar{\theta}(x, p) \times$$

$$\times \frac{\mathrm{ch}\left[\sqrt{\frac{p}{a}}(\delta - y)\right]}{\frac{\lambda}{\alpha}\sqrt{\frac{p}{a}}\,\mathrm{sh}\left[\sqrt{\frac{p}{a}}\,\delta\right] + \mathrm{ch}\left[\sqrt{\frac{p}{a}}\,\delta\right]}. \tag{6}$$

Here,

$$k = \frac{\alpha}{\gamma LC};\quad A_n = \frac{2\sin[2\mu_n]}{2\mu_n + \sin[2\mu_n]};$$

$$a_n = \frac{w}{kxA_nB_n};\quad B_n = \frac{a\mu_n^2}{\delta^2};$$

μ_n are the roots of the transcendental equation

$$\mathrm{ctg}\,\mu = \frac{1}{\mathrm{Bi}}\mu,$$

whose solution is given, for example, in [2]. The inverse transform of the n-th term of the infinite product

$$\prod_{n=1}^{\infty} \exp\left\{\frac{1}{a_n(p+B_n)}\right\}$$

is found in the form [3]

$$\theta_n(x, t) = \mathrm{L}^{-1}\left[\exp\left\{\frac{1}{a_n(p+B_n)}\right\}\right] =$$

$$= \mathrm{L}^{-1}\left[p\left(\exp\left\{\frac{1}{a_n(p+B_n)}\right\} - 1\right)\frac{1}{p} + 1\right] =$$

$$= \int_0^t \frac{\exp\{-B_n\xi\}}{\sqrt{a_n\xi}} I_1\left(2\sqrt{\frac{\xi}{a_n}}\right)d\xi + 1 =$$

$$= \sum_{k=0}^{\infty} \frac{1}{a_n^{k+1}k!\,(k+1)} \int_0^t \xi^k \exp\{-B_n\xi\}\,d\xi + 1. \tag{8}$$

Functions of this type have been investigated quite thoroughly and tabulated (see [4, 5]).

Using the rule for finding the inverse transform from a transform product [3], and denoting by S_t the operator of successive convolution of the functions θ_n with differentiation, i.e.,

$$S_t \prod_{n=1}^{2} \theta_n = \frac{d}{dt}\int_0^t \theta_2(t-\xi)\,\theta_1(\xi)\,d\xi,$$

$$S_t \prod_{n=1}^{3} \theta_n = \frac{d}{dt}\int_0^t \theta_3(t-q)\left\{\frac{d}{dq}\int_0^q \theta_2(q-\xi)\,\theta_1\,d\xi\right\}dq,$$

$$\cdots\cdots\cdots\cdots$$

$$S_t \prod_{n=1}^{\infty} \theta_n = L^{-1}\left[\prod_{n=1}^{\infty} \exp\left\{\frac{1}{a_n(p+B_n)}\right\}\right],$$

we write the solution for the temperature of the heat-transfer agent and the metal wall in the form

$$\theta(x, t) = \begin{cases} 0 \quad \text{at} \quad t < \frac{x}{w}, \\ \prod_{n=1}^{\infty} \exp\left\{-A_n\frac{kx}{w}\right\} \times \\ \times S_t \prod_{n=1}^{\infty} \theta_n\left(x, t - \frac{x}{w}\right) \quad \text{at} \quad t > \frac{x}{w}, \end{cases} \tag{10}$$

$$T(x, y, t) =$$

$$= \frac{d}{dt}\int_0^t \left[1 - \sum_{n=1}^{\infty} \Phi_n \exp\{-B_n(t-\xi)\}\right]\theta(x, \xi)\,d\xi. \tag{11}$$

If we now consider the problem with a variable temperature at the channel inlet $\theta_{in}(t)$, we can write the solution for the temperature of the heat-transfer agent and the metal in the form

$$\tilde{\theta}(x, t) = \frac{d}{dt}\int_0^t \theta(x, \xi)\,\theta_{\text{вх}}(t-\xi)\,d\xi, \tag{12}$$

$$\tilde{T}(x, y, t) =$$

$$= \frac{d}{dt}\int_0^t \left[1 - \sum_{n=1}^{\infty} \Phi_n \exp\{-B_n(t-\xi)\}\right]\tilde{\theta}(x, \xi)\,d\xi. \tag{13}$$

In (11) and (13),

$$\Phi_n = \frac{2\sin[\mu_n]\cos\left[\mu_n\frac{\delta - y}{\delta}\right]}{\mu_n + \sin[\mu_n]\cos[\mu_n]}.$$

Cylindrical symmetrical channel. We now consider a symmetrical cylindrical channel (figure, b) semi-infinite in the direction of the x-axis. The thickness of the metal wall is equal to $r_2 - r_1$.

With the same assumptions as for the plane channel, the mathematical model describing the nonsteady temperature fields in a cylindrical symmetrical channel has the form:

the equation of heat transport in the moving heat-transfer agent

$$w\,\frac{\partial\theta}{\partial x}+\frac{\partial\theta}{\partial t}=\frac{-\alpha\,2\pi r_1}{\gamma sC}\,[\theta-T|_{r=r_1}],\qquad(14)$$

the boundary and initial conditions

$$\theta|_{x=0}=1,\ \theta|_{t=0}=0,\qquad(15)$$

the heat conduction equation for the metal of the cylinder in cylindrical coordinates

$$\frac{\partial T}{\partial t}=a\,\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right),\qquad(16)$$

the boundary and initial conditions

$$-\lambda\,\frac{\partial T}{\partial r}\bigg|_{r=r_1}=\alpha\,[\theta-T|_{r=r_1}],$$

$$\frac{\partial T}{\partial r}\bigg|_{r=r_2}=0,\ T|_{t=0}=0.\qquad(17)$$

We find the solution for the temperature $\theta(x,t)$ of the heat-transfer agent and the temperature $T(x,r,t)$ of the metal.

The construction of the solution for system (14)–(17) is analogous to the case of a plane symmetrical channel; therefore, omitting the intermediate calculations, we have

$$\theta(x,\ t)=\begin{cases}0\ \text{at}\ t<\dfrac{x}{w},\\ \prod\limits_{n=1}^{\infty}\exp\left\{-D_n\,\dfrac{kx}{w}\right\}\times\\ \times\,S_t\prod\limits_{n=1}^{\infty}\vartheta_n\left(x,\ t-\dfrac{x}{w}\right)\ \text{at}\ t>\dfrac{x}{w},\end{cases}\qquad(18)$$

$$T(x,\ r,\ t)=$$
$$=\frac{d}{dt}\int\limits_0^t\left[1-\sum_{n=1}^{\infty}E_n\exp\{-F_n(t-\xi)\}\right]\theta(x,\ \xi)\,d\xi.\qquad(19)$$

In the case of a variable inlet temperature $\theta_{in}(t)$,

$$\tilde\theta(x,\ t)=\frac{d}{dt}\int\limits_0^t\theta(x,\ \xi)\,\theta'_{\text{вх}}(t-\xi)\,d\xi,\qquad(20)$$

$$\tilde T(x,\ r,\ t)=$$
$$=\frac{d}{dt}\int\limits_0^t\left[1-\sum_{n=1}^{\infty}E_n\exp\{-F_n(t-\xi)\}\right]\tilde\theta(x,\ \xi)\,d\xi.\qquad(21)$$

The following notation has been used in Eqs. (18)–(21):

$$\vartheta_n(x,\ t)=\int\limits_0^t\frac{\exp\{-F_n\xi\}}{\sqrt{g_n\xi}}\;I_1\left(2\sqrt{\frac{\xi}{g_n}}\right)d\xi+1,$$

$$F_n=\frac{\mu_n^2 a}{r_1^2},\quad g_n=\frac{w}{kxD_nF_n},$$

$$D_n=\frac{R(r_1,\ \mu_n)}{F_nZ'(\mu_n)},\ E_n=\frac{R(r,\ \mu_n)}{F_nZ'(\mu_n)},$$

$$R(r,\ \mu_n)=\mathrm{i}\ \frac{\pi}{2}\ [J_1(m\mu_n)N_0(d\mu_n)-J_0(d\mu_n)N_1(m\mu_n)],$$

$$d=\frac{r}{r_1},\quad m=\frac{r_2}{r_1},$$

$$Z'(\mu_n)=\mathrm{i}\,\frac{\pi}{2}\,\frac{1}{2a}\times$$
$$\times\left\{\frac{\lambda r_1}{\alpha}\,[J_1'(\mu_n)\,N_1(m\mu_n)-J_1(m\mu_n)\,N_1'(\mu_n)]-\right.$$
$$-\frac{\lambda r_2}{\alpha}\,[J_1'(m\mu_n)\,N_1(\mu_n)-J_1(\mu_n)\,N_1'(m\mu_n)]+$$
$$+\left(\frac{r_1^2}{\mu_n}-\frac{\lambda}{\alpha}\,\frac{r_1}{\mu_n}\right)\times$$
$$\times[J_1(m\mu_n)\,N_1(\mu_n)-J_1(\mu_n)\,N_1(m\mu_n)]-$$
$$\left.-\frac{r_1r_2}{\mu_n}\,[J_1'(m\mu_n)\,N_0(\mu_n)-J_0(\mu_n)\,N_1'(m\mu_n)]\right\},$$

$$2J_1'(m\mu_n)=J_0(m\mu_n)-J_2(m\mu_n),$$
$$2J_1'(\mu_n)=J_0(\mu_n)-J_2(\mu_n),$$
$$2N_1'(\mu_n)=N_0(\mu_n)-N_2(\mu_n),$$
$$2N_1'(m\mu_n)=N_0(m\mu_n)-N_2(m\mu_n).\qquad(22)$$

μ_n are the roots of the following transcendental equation:

$$\frac{J_0(\mu)\,N_1(m\mu)-N_0(\mu)\,J_1(m\mu)}{J_1(\mu)\,N_1(m\mu)-N_1(\mu)\,J_1(m\mu)}=-\frac{1}{\mathrm{Bi}}\ \mu,\qquad(23)$$

here,

$$\mu=\mathrm{i}\sqrt{\frac{p}{a}}\,r_1;\ \mathrm{Bi}=\frac{\alpha r_1}{\lambda}.$$

The given equation has a denumerable set of positive roots

$$0<\mu_1<\mu_2<\ldots<\mu_n<\ldots$$

The first approximate value of μ_n can be determined graphically, for example. The zeros on the left-hand side of Eq. (23) are determined from the solution of the equation

$$J_0(\mu)\,N_1(m\mu)-N_0(\mu)\,J_1(m\mu)=0,\qquad(24)$$

and the points at which the left-hand side goes to infinity from the solution of the equation

$$J_1(\mu)\,N_1(m\mu)-N_1(\mu)\,J_1(m\mu)=0.\qquad(25)$$

The first six roots of Eqs. (24) and (25) for various m are presented in [6].

The form of solutions (10)–(13) and (18)–(21) is quite complicated. This applies particularly to the construction of the functions $S_t\prod\limits_{n=1}^{\infty}\theta_n$ and $S_t\prod\limits_{n=1}^{\infty}\vartheta_n$, when it is necessary to carry out convolution of the functions θ_n or ϑ_n with subsequent differentiation. For the beginning of the transient process, this complexity can be eliminated by using the asymptotic representation of a Bessel function of imaginary argument I_ν in the neighborhood of zero [6]:

$$I_1\left(2\sqrt{\frac{t}{a_n}}\right)\simeq\sqrt{\frac{t}{a_n}}, \tag{26}$$

then

$$\theta_n(x, t)\simeq\int_0^t \frac{\exp\{-B_n\xi\}}{\sqrt{a_n\xi}}\sqrt{\frac{\xi}{a_n}}\,d\xi+1=$$
$$=1+\frac{kx}{w}A_n[1-\exp\{-B_n t\}]. \tag{27}$$

The approximate form of the functions $\theta_n(x, t)$ obtained makes it possible to perform the operations of integration and differentiation quite simply, the region of application of formula (27) being determined by the region of applicability of expression (26).

In conclusion, we note that if we content ourselves with the first terms of the expansion, i.e., if we write the solutions in the form (e.g., for the case of a plane channel and unit perturbation at the inlet)

$$\theta(x, t)=\begin{cases}0 \text{ at } t<\dfrac{x}{w},\\ \exp\left\{-A_1\dfrac{kx}{w}\right\}\theta_1\left(x, t-\dfrac{x}{w}\right) \text{ at } t>\dfrac{x}{w},\end{cases} \tag{28}$$

$$T(x, y, t)=$$
$$=\frac{d}{dt}\int_0^t[1-\Phi_1\exp\{-B_1(t-\xi)\}]\,\theta(x, \xi)\,d\xi; \tag{29}$$

we see that, physically, this corresponds to the case of a thin, highly heat-conducting channel wall. Similar problems were examined, for example, in [7, 8, 9].

NOTATION

θ_{in} is the temperature of the heat-transfer agent at the channel inlet; θ and T are, respectively, the temperatures of the heat-transfer agent and of the metal for unit jump in inlet temperature; $\theta^{\sim}$ and $T^{\sim}$ are the same for an arbitrary change in inlet temperature; Bi is the Biot number; α is the coefficient of heat transfer from heat-transfer agent to channel wall; C, γ, and w are, respectively, the specific heat, specific weight, and velocity of the heat-transfer agent; λ and a are, respectively, the thermal conductivity and thermal diffusivity of the metal; t is the time; x, y, and z are coordinate axes; r is the variable radius; r_1 and r_2 are, respectively, the inside and outside radii of the cylinder; s is the cross-sectional area of the heat-transfer agent in the cylindrical channel; 2L is the width of the plane channel; δ is the thickness of the plane channel wall; L^{-1} is the Laplace-Carson inverse transform operator; $J_\nu(x)$, $N_\nu(x)$, $I_\nu(x)$ are the cylindrical Bessel and Neumann functions and the Bessel function of imaginary argument, respectively.

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